Int. J. Solids Structures Vol. 29, No. 23, pp. 2907–2919, 1992
Printed in Great Britain.

0020–7683/92 $5.00 + .00

ON THE THERMOMECHANICS OF COMPOSITES WITH IMPERFECTLY BONDED INTERFACES AND DAMAGE

G. J. DVORAK
Institute Center for Composite Materials and Structures, Rensselaer Polytechnic Institute,
Troy, NY 12180, U.S.A.

and

Y. BENVENISTE†
Department of Solid Mechanics, Materials and Structures, Faculty of Engineering,
Tel-Aviv University, Ramat-Aviv, 69978, Israel

(*Received* 29 *July* 1991; *in revised form* 19 *March* 1992)

Abstract—General connections are established between the mechanical and thermal responses of composite materials with debonded or imperfectly bonded interfaces, and with internal cracks or cavities. In particular, such results are found for multiphase composites or polycrystals in which normal and/or shear displacement jumps may exist at interfaces or cracks, consistent with complete debonding or with the presence of a nonlinearly elastic interphase layer. In two-phase systems with isotropic phases and sliding interfaces, we also recover exact connections between the mechanical and thermal stress or strain fields in the phases.

1. INTRODUCTION

Evaluation of thermoelastic properties of composite materials is of considerable interest, particularly in high-temperature ceramic systems. Although perfect bonding between the phases may be desirable, various types of imperfect bonding at interfaces, as well as internal cracking may exist in actual systems. Any such damage mode will cause a change in overall stiffness, in local mechanical fields, and also in the overall thermal expansion coefficients and in the thermal stress and strain fields. It is well known that in perfectly bonded systems, the overall thermal properties can be evaluated in terms of phase properties and mechanical concentration factors (Levin, 1967). More general relations involving local fields also exist for certain perfectly bonded two-phase systems (Dvorak, 1983, 1986, 1990; Dvorak and Chen, 1989; Benveniste and Dvorak, 1990a), and also for two-phase composites with isotropic constituents and slipping interfaces (Benveniste and Dvorak, 1990b).

The present paper extends this line of inquiry, and establishes such connections for many other damaged composite materials. In particular, we show in the first part of the paper that the Levin-type connections are recovered in multiphase composite systems of arbitrary phase geometry and material symmetry, even if the interfaces, or their parts, undergo debonding which is either complete, or consistent with the presence of a very thin nonlinearly elastic interphase layer which permits both normal and shear displacement jumps at interfaces. In the second part, special forms of these results are found for two-phase composites. Moreover, in two-phase systems with isotropic constituents and slipping interfaces, exact relationships are found between mechanical and thermal stress or strain fields in the phases. This is accomplished with the help of uniform strain and stress fields in heterogeneous media (Dvorak, 1990; Benveniste and Dvorak, 1990a).

The emphasis is on evaluation of general thermomechanical connections rather than the formulation of micromechanical models. Examples of the latter may be found in other recent references, e.g. Chen and Argon (1979a, b); Lené and Leguillon (1982); Benveniste and Aboudi (1984); Mura *et al.* (1985); Benveniste and Miloh (1986); Tsuchida *et al.* (1986); Jasiuk *et al.* (1988); Achenbach and Zhu (1989); Jasiuk and Tong (1989); Hashin (1990). Therefore, throughout the paper we assume that the local fields caused by

---

† Also Visiting Professor at RPI.

mechanical loads can be evaluated by an independent analysis. Our purpose is to provide a general methodology for evaluation of the thermal response of damaged composites from the various solutions of mechanical loading problems.

2. MULTIPHASE COMPOSITES

2.1. *Phase and interface properties*

We first consider multiphase media with N constituent phases, which may represent such actual systems as matrix-based composites or polycrystals, and focus our attention at a sufficiently large representative volume which has the same effective properties as any other volume of such or larger size. If a matrix is present, then it will be denoted by $r = 1$, and $r = 2, 3, \ldots, N$ will represent the reinforcing phases. All phases are linear thermoelastic solids, their constitutive relations are

$$\boldsymbol{\sigma}_r = \mathbf{L}_r \boldsymbol{\varepsilon}_r + \mathbf{l}_r \theta_0, \quad \boldsymbol{\varepsilon}_r = \mathbf{M}_r \boldsymbol{\sigma}_r + \mathbf{m}_r \theta_0, \quad r = 1, 2, \ldots, N. \tag{1}$$

where $\boldsymbol{\sigma}_r$, $\boldsymbol{\varepsilon}_r$, $\mathbf{L}_r$, $\mathbf{l}_r$, θ_0 denote, respectively, the stress, strain, stiffness, thermal stress tensors and a uniform temperature change. $\mathbf{M}_r = \mathbf{L}_r^{-1}$ and $\mathbf{m}_r = -\mathbf{M}_r \mathbf{l}_r$ are the compliance and thermal strain tensors.

Damage in composites may be due to internal cavities or cracks, and imperfect bonding between the phases. Imperfect bonding may be regarded in terms of a thin interphase region of certain stiffness, or as interface cracks and cavities. The interface between phases r and s will be represented in this paper by an idealized geometrical surface of zero thickness. Nevertheless, it will be convenient to think of these interfaces as two-sided surfaces S_{rs} and S_{sr} adjacent to phases r and s respectively; such a notation will also help symmetrize many expressions in the paper. The displacement field may or may not be discontinuous across such interfaces. Should a cavity or a crack develop between the phases r and s, the surface of that vacuous zone will be denoted by S_{rr} and S_{ss}. The surface S_{rr} will be that in contact with phase r, and S_{ss} that in contact with phase s, (see Fig. 1a). The phases may also contain internal cracks or cavities. The surface of such a defect which is internal to phase r will be

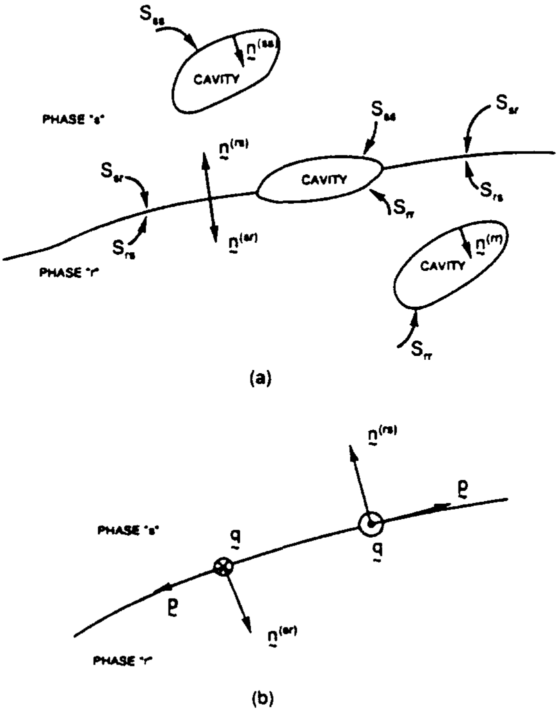

Fig. 1. (a) Interface between two phases r and s. (b) Possible choice of coordinate systems at interfaces.

denoted by S_{rr}. In the case of a thin crack in phase r, it may be convenient, though not necessary, to consider the decomposition $S_{rr} = S_{rr}^{+} \cup S_{rr}^{-}$ where S_{rr}^{+} and S_{rr}^{-} denote the upper and lower surfaces of the crack.

At any point on the interfacial surface between phase r and phase s, it will be convenient to define the unit normals $\mathbf{n}^{(sr)} = -\mathbf{n}^{(rs)}$ from phase s to phase r. At the surface of a cavity or crack which is in contact with phase r we will define the normal $\mathbf{n}^{(rr)}$ from phase r into the vacuous zone.

The displacements and tractions, together with the unit normals described above at any point $\mathbf{x}$ of the interfacial surface, will be described in a *single* Cartesian coordinate system. This Cartesian system can in principle be fixed in space, but can also be conveniently chosen at the generic point $\mathbf{x}$ on the interface. In the latter alternative, we may choose either $(\mathbf{n}^{(rs)}, \mathbf{p}, \mathbf{q})$ *or* $(\mathbf{n}^{(sr)}, \mathbf{p}, \mathbf{q})$ where $\mathbf{p}$ and $\mathbf{q}$ describe the tangential unit vectors at the interface (see Fig. 1b). For a cavity or a crack, we will choose $(\mathbf{n}^{(rr)}, \mathbf{p}, \mathbf{q})$. With no loss of generality, we thus adopt the coordinate system $(\mathbf{n}^{(rs)}, \mathbf{p}, \mathbf{q})$, where in the case of a pore or a crack there is $r = s$.

At any generic point $\mathbf{x}$ of the interface, let us define the traction vector exerted from phase r to phase s as $\mathbf{t}^{(rs)}$, and from s to r as $\mathbf{t}^{(sr)}$:

$$\mathbf{t}^{(rs)} = (t_n^{(rs)}, t_p^{(rs)}, t_q^{(rs)})^{\mathrm{T}}, \quad \mathbf{t}^{(sr)} = (t_n^{(sr)}, t_p^{(sr)}, t_q^{(sr)})^{\mathrm{T}}. \tag{2}$$

We note that both are expressed in the coordinate system $(\mathbf{n}^{(rs)}, \mathbf{p}, \mathbf{q})$. Regardless of the nature of the bond, $\mathbf{t}^{(rs)}$ must be in equilibrium with $\mathbf{t}^{(sr)}$, thus

$$\mathbf{t}^{(rs)} + \mathbf{t}^{(sr)} = 0. \tag{3}$$

For a generic point $\mathbf{x}$, at a surface S_{rr} of a cavity or crack adjacent to phase r, it follows that

$$\mathbf{t}^{(rr)} = (t_n^{(rr)}, t_p^{(rr)}, t_q^{(rr)})^{\mathrm{T}} = 0, \tag{4}$$

where the quantities are described in the coordinate system $(\mathbf{n}^{(rr)}, \mathbf{p}, \mathbf{q})$ defined above.

Displacement vectors at any point $\mathbf{x}$ of the interface are defined at each side and expressed in the coordinate system $(\mathbf{n}^{(rs)}, \mathbf{p}, \mathbf{q})$ as:

$$\mathbf{u}^{(r)} = (u_n^{(r)}, u_p^{(r)}, u_q^{(r)})^{\mathrm{T}}, \quad \mathbf{u}^{(s)} = (u_n^{(s)}, u_p^{(s)}, u_q^{(s)})^{\mathrm{T}}, \tag{5}$$

the difference or jump in those displacements across the interface will be denoted by

$$[\mathbf{u}] = \mathbf{u}^{(r)} - \mathbf{u}^{(s)}. \tag{6}$$

These conventions permit us to define the following types of interface bonding that will be of interest in the sequel. A *perfectly bonded* interface which does not contain any interphase layer is characterized by the relations

$$\mathbf{t}^{(rs)} = -\mathbf{t}^{(sr)} \neq \mathbf{0}, \quad [\mathbf{u}] = \mathbf{0}. \tag{7}$$

At a *debonded interface* which is actually considered a cavity or a crack,

$$\mathbf{t}^{(rr)} = \mathbf{0}. \tag{8}$$

Our interest will frequently focus on *imperfectly bonded interfaces*, which allow nonvanishing relative displacements to exist together with nonzero tractions. The implication is that the displacements and tractions are related in a certain way at each instant of loading, as if the interfaces were connected by a very thin layer of interphase material. We limit our attention to systems where such relationships, or the properties of the interphase, are described by the incremental form

$$\begin{aligned} \mathrm{d}[u_n] &= M_{nn}(\mathbf{t}^0)\,\mathrm{d}t_n, \\ \mathrm{d}[u_p] &= M_{pp}(\mathbf{t}^0)\mathrm{d}t_p + M_{pq}(\mathbf{t}^0)\mathrm{d}t_q, \\ \mathrm{d}[u_q] &= M_{qp}(\mathbf{t}^0)\mathrm{d}t_p + M_{qq}(\mathbf{t}^0)\mathrm{d}t_q, \end{aligned} \tag{9}$$

at each current magnitude $\mathbf{t}^{(sr)} = -\mathbf{t}^{(rs)} = \mathbf{t}^0$ of the interface traction; for simplicity we have denoted $(\mathrm{d}t_n, \mathrm{d}t_p, \mathrm{d}t_q) = (\mathrm{d}t_n^{(sr)}, \mathrm{d}t_p^{(sr)}, \mathrm{d}t_q^{(sr)})$. The $M_{\alpha\beta}$, with α, β = n, p, q, are the instantaneous "compliances" of the interface, or interphase layer, and are assumed to be represented by smooth, continuous functions, that satisfy the symmetry condition $M_{\alpha\beta} = M_{\beta\alpha}$. Since the interphase is assumed to be very thin, the contributions of the terms $M_{np}\mathrm{d}t_p$, $M_{nq}\mathrm{d}t_q$, etc., to d[**u**] are considered to be insignificant and are neglected.

The imperfectly bonded interface that can be represented by (9) includes nonlinearly elastic coatings, and also interfaces which are weak in shear but perfectly bonded in the normal direction, in which case $M_{nn}(\mathbf{t}^0) = 0$, and $[u_n] = 0$. The representation (9) may imply an interpenetration in the normal displacement components u_n across the idealized interface in the case of a normal compressive traction. However, since we limit ourselves to small strains, and since these interfaces do in fact represent interphase regions with a certain thickness, such interpenetration can be accommodated by compression of the interphase. Interfaces that exhibit frictional contact, perfect bonding, or complete debonding are not represented by (9). Indeed, interface friction would relate the tangential components of the traction to the compressive normal component when $[u_n] = 0$, but without reference to the magnitude of [**u**], although the ratio of $t_p^{(rs)}$ to $t_q^{(rs)}$ may determine the direction of [**u**].

2.2. *Local fields*

Let a representative volume of a composite material be subjected to an overall uniform stress $\bar{\sigma}$, or strain $\bar{\varepsilon}$, and to a uniform temperature change θ_0. In particular, we select the overall thermomechanical loading on external surface S as

$$\mathbf{u}(S) = \varepsilon_0\mathbf{x}, \quad \theta(S) = \theta_0, \tag{10}$$

so that $\bar{\varepsilon} = \varepsilon_0$, and examine its effect on local strain and displacement fields in the phase.

We assume that the local fields can be evaluated by an independent analysis of each specific system. Examples can be found in the references listed in the Introduction. In systems which undergo progressive debonding, i.e. involving changes in the size or location of the interfaces (8), and progressive deformation induced at the imperfectly bonded interfaces (9), such analysis may need to be performed at many different points of the prescribed loading path leading to the current state (10). In any event, the current local strain and displacement fields can be denoted by

$$\varepsilon_r(\mathbf{x}) = \varepsilon_r^0(\mathbf{x}\,;\varepsilon_0,\theta_0), \quad \mathbf{u}_r(\mathbf{x}) = \mathbf{u}_r^0(\mathbf{x}\,;\varepsilon_0,\theta_0), \quad \theta(\mathbf{x}) = \theta_0. \tag{11}$$

The displacement and temperature increments on the outer surface S of the representative volume are incrementally specified for a change in the temperature or strain, as

$$\mathbf{u}(S) = \varepsilon_0\mathbf{x}, \quad \theta(S) = \theta_0 + \mathrm{d}\theta_0 \tag{12}$$

or

$$\mathbf{u}(S) = (\varepsilon_0 + \mathrm{d}\varepsilon_0)\mathbf{x}, \quad \theta(S) = \theta_0. \tag{13}$$

The resulting incremental strain and displacement fields to be superimposed on (11) are:

$$\mathrm{d}\boldsymbol{\varepsilon}_r(\mathbf{x}) = \mathbf{a}_r(\mathbf{x}\,;\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\theta_0, \quad \mathrm{d}\mathbf{u}_r = \mathbf{d}_r(\mathbf{x}\,;\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\theta_0 \tag{14}$$

or

$$\mathrm{d}\boldsymbol{\varepsilon}_r(\mathbf{x}) = \mathbf{A}_r(\mathbf{x}\,;\boldsymbol{\varepsilon}_0,\theta_0)\mathrm{d}\boldsymbol{\varepsilon}_0, \quad \mathrm{d}\mathbf{u}_r = \mathbf{D}_r(\mathbf{x}\,;\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\boldsymbol{\varepsilon}_0, \tag{15}$$

where $\mathbf{a}_r(\mathbf{x}\,;\,\boldsymbol{\varepsilon}_0,\,\theta_0)$, $\mathbf{d}_r(\mathbf{x}\,;\,\boldsymbol{\varepsilon}_0,\,\theta_0)$ are certain thermal influence functions, and $\mathbf{A}_r(\mathbf{x}\,;\,\boldsymbol{\varepsilon}_0,\,\theta_0)$, $\mathbf{D}_r(\mathbf{x}\,;\,\boldsymbol{\varepsilon}_0,\,\theta_0)$ are the mechanical influence functions. Their dependence on $\boldsymbol{\varepsilon}_0$ and θ_0 is the consequence of possible progressive debonding and/or nonlinear behavior of the interfaces.

2.3. *Overall properties*

The overall average strain in the presence of imperfect bonding is the sum of average phase strains, and strains that may be contributed by the relative displacement at the interfaces as well as by the presence of cavities and cracks. A derivation for two-phase, matrix-based composites has been given by Benveniste (1985). Here we present a more general result that applies to multiphase composites, not necessarily matrix-based, which may contain cracks and cavities.

Using the notation introduced in Section 2.1, we show in Appendix A that the average strain in such a composite is given by

$$\bar{\boldsymbol{\varepsilon}} = \sum_{r=1}^{N} c_r \bar{\boldsymbol{\varepsilon}}^{(r)} - \sum_{r=1}^{N}\sum_{s=1}^{N} \mathbf{J}_{rs}, \tag{16}$$

where c_r denotes the volume fraction of phase r, N is the number of phases, $\bar{\boldsymbol{\varepsilon}}^{(r)}$ is the average strain within that phase, and the second order tensors $\mathbf{J}_{rs}$ are given by:

$$J_{ij}^{(rs)} = \frac{1}{2V}\int_{S_{rs}} (u_i^{(r)}n_j^{(rs)} + u_j^{(r)}n_i^{(rs)})\,\mathrm{d}S_{rs}, \quad J_{ij}^{(sr)} = \frac{1}{2V}\int_{S_{sr}} (u_i^{(s)}n_j^{(sr)} + u_j^{(r)}n_i^{(sr)})\,\mathrm{d}S_{sr}. \tag{17}$$

It is noted here that thinking of the interface surface between the phases r and s as two-sided surfaces S_{rs} and S_{sr} allows a symmetrical representation of eqns $(17)_1$ and $(17)_2$.

It is often convenient to introduce concentration factors that reflect the presence of damage. In particular, under the load increments prescribed in (12) and (13), one finds from (14), (15) and (17):

$$\mathrm{d}\mathbf{J}_{rs} = \mathbf{F}_{rs}(\boldsymbol{\varepsilon}_0,\theta_0)\mathrm{d}\boldsymbol{\varepsilon}_0 + \mathbf{f}_{rs}(\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\theta_0, \quad r,s = 1,2,\ldots,N, \tag{18}$$

where the concentration factor tensors $\mathbf{F}_{rs}$ and $\mathbf{f}_{rs}$ are related to the $\mathbf{D}_r$ and $\mathbf{d}_r$ influence functions in $(14)_2$ and $(15)_2$ as:

$$F_{ijkl}^{rs} = \frac{1}{2V}\int_{S_{rs}} (D_{ikl}^{(r)}(\mathbf{x})n_j^{(rs)} + D_{jkl}^{(r)}(\mathbf{x})n_i^{(rs)})\,\mathrm{d}S_{rs},$$

$$f_{ij}^{rs} = \frac{1}{2V}\int_{S_{rs}} (\mathbf{d}_i^{(r)}(\mathbf{x})n_j^{(rs)} + d_j^{(r)}(\mathbf{x})n_i^{(rs)})\,\mathrm{d}S_{rs}. \tag{19}$$

The concentration factors $\mathbf{F}_{sr}$ and $\mathbf{f}_{sr}$ related to $\mathrm{d}\mathbf{J}_{sr}$ are described simply by interchanging r and s in (18) and (19). Together with the related factors defined in (14) and (15), they facilitate the description of overall properties of the damaged composite materials. We refer again to the representative volume of a composite material which is subjected to overall uniform stress $\bar{\boldsymbol{\sigma}}$, or strain $\bar{\boldsymbol{\varepsilon}}$ and to a uniform temperature change θ_0. Since the overall response is not necessarily linear, it is sought in the incremental form

$$d\bar{\boldsymbol{\sigma}} = \mathbf{L}(\bar{\boldsymbol{\varepsilon}}, \theta_0)\,d\bar{\boldsymbol{\varepsilon}} + \mathbf{l}(\bar{\boldsymbol{\varepsilon}}, \theta_0)\,d\theta_0,$$
$$d\bar{\boldsymbol{\varepsilon}} = \mathbf{M}(\bar{\boldsymbol{\varepsilon}}, \theta_0)\,d\bar{\boldsymbol{\sigma}} + \mathbf{m}(\bar{\boldsymbol{\varepsilon}}, \theta_0)\,d\theta_0, \tag{20}$$

where $\mathbf{L}(\bar{\boldsymbol{\varepsilon}}, \theta_0)$ and $\mathbf{l}(\bar{\boldsymbol{\varepsilon}}, \theta_0)$ are the instantaneous stiffness and thermal stress tensors which depend on the current overall strain and temperature. The $\mathbf{M}(\bar{\boldsymbol{\varepsilon}}, \theta_0)$ and $\mathbf{m}(\bar{\boldsymbol{\varepsilon}}, \theta_0)$ are the corresponding compliance and thermal strain tensors.

These effective properties can be determined once the concentration factors $\mathbf{a}_r$, $\mathbf{A}_r$, the volume averages of the influence functions $\mathbf{a}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)$, $\mathbf{A}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)$ introduced in (14) and (15), and the tensors $\mathbf{F}_{rs}$, $\mathbf{f}_{rs}$ defined in (19) are known. Equations $(14)_1$, $(15)_1$, (16), (18) and $(20)_2$ readily provide the following expressions for $\mathbf{L}$ and $\mathbf{l}$:

$$\mathbf{L}(\boldsymbol{\varepsilon}_0, \theta_0) = \mathbf{L}_1 + \sum_{r=2}^{N} c_r(\mathbf{L}_r - \mathbf{L}_1)\mathbf{A}_r(\boldsymbol{\varepsilon}_0, \theta_0) + \mathbf{L}_1 \sum_{r=1}^{N}\sum_{s=1}^{N} \mathbf{F}_{rs}(\boldsymbol{\varepsilon}_0, \theta_0)$$
$$\mathbf{l}(\boldsymbol{\varepsilon}_0, \theta_0) = \sum_{r=1}^{N} c_r\mathbf{l}_r + \sum_{r=2}^{N} c_r(\mathbf{L}_r - \mathbf{L}_1)\mathbf{a}_r(\boldsymbol{\varepsilon}_0, \theta_0) + \mathbf{L}_1 \sum_{r=1}^{N}\sum_{s=1}^{N} \mathbf{f}_{rs}(\boldsymbol{\varepsilon}_0, \theta_0). \tag{21}$$

Similar equations can be obtained for $\mathbf{M}$ and $\mathbf{m}$.

2.4. *Evaluation of* l *and* m

In his (1967) paper, Levin found an expression which relates the thermal stress tensor $\mathbf{l}$ to the mechanical concentration factors $\mathbf{A}_r$ of the phases and to phase thermal vectors $\mathbf{l}_r$ in an undamaged composite with perfectly bonded interfaces. An analogous relation exists between the overall thermal strain tensor $\mathbf{m}$ and the stress concentration factor $\mathbf{B}_r$ and phase thermal strain tensors $\mathbf{m}_r$. Under certain conditions, a similar formula can be derived for composites with imperfectly bonded or partially debonded interfaces defined in (7)–(9). The derivation presented here will use the reciprocal theorem, although a similar result follows from a modified principle of virtual work for composites of this type (Benveniste, 1985). For completeness, we present in Appendix B a derivation of the reciprocal theorem which accounts for the effect of applied eigenstrain fields and imperfect interfaces.

Suppose that the composite has been loaded to some current known state $(\boldsymbol{\varepsilon}_0, \boldsymbol{\sigma}_0, \theta_0)$, where the extent of partial and/or complete interface debonding has been evaluated such that all coefficients in (9) and the mechanical influence functions $\mathbf{A}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)$, $\mathbf{B}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)$† and $\mathbf{D}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)$ in (15) are known together with the instantaneous overall stiffness $\mathbf{L}$ and compliance $\mathbf{M} = \mathbf{L}^{-1}$. In this current state, we apply two separate load increments (′) and (″) such that there is no change in the type of interface bonding (7)–(9) on S_{rs}. First, an overall uniform stress increment $d\boldsymbol{\sigma}_0$ is applied at the current temperature $\theta(S) = \theta_0$. According to (15), this will cause the strain and displacement fields in the phases

$$d\boldsymbol{\varepsilon}'_r(\mathbf{x}) = \mathbf{M}_r\mathbf{B}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)\,d\boldsymbol{\sigma}'_0 = \mathbf{A}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)\mathbf{M}\,d\boldsymbol{\sigma}'_0, \tag{22}$$

$$d\mathbf{u}'_r(\mathbf{x}) = \mathbf{D}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)\mathbf{M}\,d\boldsymbol{\sigma}'_0. \tag{23}$$

Next, the overall temperature is changed from θ_0 to θ''_0 at fixed overall stress $\boldsymbol{\sigma}_0$. This will cause the local thermal strains $\mathbf{m}_r\,d\theta''$ which can be expressed as

$$d\boldsymbol{\lambda}''_r = -\mathbf{L}_r\mathbf{m}_r\,d\theta''_0 = \mathbf{l}_r\,d\theta''_0, \tag{24}$$

as well as the displacement fields denoted by

---

† The influence function $\mathbf{B}_r(\mathbf{x}; \boldsymbol{\varepsilon}_0, \theta_0)$ relates the stress increment $d\boldsymbol{\sigma}_0$ to the local field $d\boldsymbol{\sigma}_r$ in the same way as the function $\mathbf{A}_r$ relates the strains in (15).

$$\mathrm{d}\mathbf{u}_r'' = \mathbf{d}_r(\mathbf{x}:\boldsymbol{\varepsilon}_0, \theta_0)\,\mathrm{d}\theta_0''. \tag{25}$$

Let us now use the reciprocal theorem given in (B8) in its incremental form. Note that $\mathrm{d}F_i' = \mathrm{d}F_i'' = \mathrm{d}\lambda_{ij}' = \mathrm{d}t_i'' = 0$, and write

$$\int_S \mathrm{d}\boldsymbol{\sigma}_0'\mathbf{n}\cdot\mathrm{d}\mathbf{u}''\,\mathrm{d}S + \int_{S_{\mathrm{int}}} \mathrm{d}\mathbf{t}_{\mathrm{int}}'\cdot[\mathrm{d}\mathbf{u}'']\,\mathrm{d}S_{\mathrm{int}} = \sum_{r=1}^{N}\int_{V_r} -\mathrm{d}\boldsymbol{\lambda}_r''\,\mathrm{d}\boldsymbol{\varepsilon}_r'\,\mathrm{d}V + \int_{S_{\mathrm{int}}} \mathrm{d}\mathbf{t}_{\mathrm{int}}''\cdot[\mathrm{d}\mathbf{u}']\,\mathrm{d}S_{\mathrm{int}}, \tag{26}$$

where we have used the notation S_{int} to denote *all* interfaces between the phases r in volumes V_r. Of course, at surfaces in contact with vacuous zones, the tractions and thus these integrals vanish.

The first integral on the left-hand side is, by definition, the scalar product of the overall stress increment with the strain increment $\mathrm{d}\sigma_{ij}^{0'}\,\mathrm{d}\bar{\varepsilon}_{ij}''$. A substitution from (9) reveals that the two integrals over S_{int} contain terms $\mathrm{d}\mathbf{t}'\,\mathbf{M}_{\mathrm{int}}\,\mathrm{d}\mathbf{t}''$ and $\mathrm{d}\mathbf{t}''\,\mathbf{M}_{\mathrm{int}}^{\mathrm{T}}\,\mathrm{d}\mathbf{t}' = \mathrm{d}\mathbf{t}'\,\mathbf{M}_{\mathrm{int}}^{\mathrm{T}}\,\mathrm{d}\mathbf{t}''$, respectively. Since (9) was assumed to admit only interfaces where $\mathbf{M}_{\mathrm{int}} = \mathbf{M}_{\mathrm{int}}^{\mathrm{T}}$, those integrals are equal and cancel each other. At locations where the interface is perfectly bonded ($[\mathrm{d}\mathbf{u}] = \mathbf{0}$), or completely debonded ($\mathrm{d}\mathbf{t} = \mathbf{0}$), both integrals vanish.

The remaining two integrals over V are rewritten with the help of (22)–(24). One form is

$$\int_V \mathrm{d}\boldsymbol{\sigma}_0'\mathbf{m}\,\mathrm{d}\theta_0''\,\mathrm{d}V = \sum_{r=1}^{N}\left\{\int_{V_r} \mathbf{L}_r\mathbf{m}_r\mathbf{M}_r\mathbf{B}_r(\mathbf{x}:\boldsymbol{\sigma}_0, \theta_0)\,\mathrm{d}\boldsymbol{\sigma}_0'\,\mathrm{d}\theta_0''\,\mathrm{d}V\right\}. \tag{27}$$

Since $\mathbf{M}_r = \mathbf{M}_r^{\mathrm{T}} = \mathbf{L}_r^{-1}$, the right-hand side integrand can be shown to be rewritten as $\mathrm{d}\boldsymbol{\sigma}_0'\mathbf{B}_r^{\mathrm{T}}\mathbf{m}_r\,\mathrm{d}\theta_0''$. Thus (27) can be solved for the overall thermal strain tensor $\mathbf{m}$ as

$$\mathbf{m}(\boldsymbol{\sigma}_0, \theta_0) = \sum_{r=1}^{N}\left\{\int_{V_r} \mathbf{B}_r^{\mathrm{T}}(\mathbf{x}:\boldsymbol{\sigma}_0, \theta_0)\mathbf{m}_r\,\mathrm{d}V\right\}. \tag{28}$$

An analogous analysis yields the expression for the overall thermal stress vector

$$\mathbf{l}(\boldsymbol{\varepsilon}_0, \theta_0) = \sum_{r=1}^{N}\left\{\int_{V_r} \mathbf{A}_r^{\mathrm{T}}(\mathbf{x}:\boldsymbol{\varepsilon}_0, \theta_0)\mathbf{l}_r\,\mathrm{d}V\right\}. \tag{29}$$

Taking the phase volume averages of the influence functions over V_r gives

$$\mathbf{m}(\boldsymbol{\sigma}_0, \theta_0) = \sum_{r=1}^{N} c_r\mathbf{B}_r^{\mathrm{T}}(\boldsymbol{\sigma}_0, \theta_0)\mathbf{m}_r, \quad \mathbf{l}(\boldsymbol{\sigma}_0, \theta_0) = \sum_{r=1}^{N} c_r\mathbf{A}_r^{\mathrm{T}}(\boldsymbol{\varepsilon}_0, \theta_0)\mathbf{l}_r. \tag{30}$$

This result is formally identical to that found by Levin (1967), however, the mechanical concentration factors entering here are those of the damaged composite, and as such they depend on the current geometry of the imperfectly bonded or debonded interfaces. Therefore, (28)–(30) should be utilized in conjunction with an incremental solution of a thermomechanical loading problem for the damaged composite material. Of course, such a solution may provide the overall strains, and (30) can then identify the purely thermal contribution. However, if the geometry changes cease at a certain load level, e.g. because the imperfectly bonded interfaces have separated, then the mechanical concentration factors remain independent of further load or temperature changes. Once these become known from the solution of a mechanical loading problem for the damaged composite, the above relations can be used to find the overall thermal properties.

3. TWO-PHASE COMPOSITES

3.1. *Overall properties*

First, consider some specific forms of the above results which apply to two-phase composite systems. Suppose that $r = 1$ denotes the matrix and $r = 2$ a reinforcing phase. Then the general expressions (21) for the overall stiffness $\mathbf{L}$ can be rearranged as:

$$\mathbf{L}(\boldsymbol{\varepsilon}^0,\theta_0) = \mathbf{L}_1 + c_2(\mathbf{L}_2-\mathbf{L}_1)\mathbf{A}_2(\boldsymbol{\varepsilon}_0,\theta_0) + \mathbf{L}_1\mathbf{A}(\boldsymbol{\varepsilon}_0,\theta_0),$$
$$\mathbf{l}(\boldsymbol{\varepsilon}^0,\theta_0) = c_1\mathbf{l}_1 + c_2\mathbf{l}_2 + c_2(\mathbf{L}_2-\mathbf{L}_1)\mathbf{a}_2(\boldsymbol{\varepsilon}_0,\theta_0) + \mathbf{L}_1\mathbf{a}(\boldsymbol{\varepsilon}_0,\theta_0), \qquad (31)$$

where the $\mathbf{A}$ and $\mathbf{a}$ tensors reflect the effect of damage, and the $\mathbf{A}_2$, $\mathbf{a}_2$ are the mechanical concentration factor tensors of the damaged composite. The overall average strain (16) now becomes

$$\bar{\boldsymbol{\varepsilon}} = c_1\bar{\boldsymbol{\varepsilon}}_1 + c_2\bar{\boldsymbol{\varepsilon}}_2 - \mathbf{J}, \qquad (32)$$

where $\bar{\boldsymbol{\varepsilon}}_r$ are the average strains in the constituents, and $\mathbf{J}$ is given by the double sum in (16) taken over $r, s = 1, 2$. From the above representation, it is seen that

$$\mathrm{d}\mathbf{J} = \mathbf{A}(\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\boldsymbol{\varepsilon}_0 + \mathbf{a}(\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\theta_0. \qquad (33)$$

For two-phase composites, an alternative expression for the $\mathbf{l}(\boldsymbol{\varepsilon}_0,\theta_0)$ in $(31)_2$ can be obtained as follows. First write (32) in incremental form, and recall that under (12) and (13) $\mathrm{d}\bar{\boldsymbol{\varepsilon}} = \mathrm{d}\boldsymbol{\varepsilon}^0$. Next, make use of $(14)_1$, $(15)_1$ and (33) to obtain

$$c_1\mathbf{A}_1(\boldsymbol{\varepsilon}_0,\theta_0) + c_2\mathbf{A}_2(\boldsymbol{\varepsilon}_0,\theta_0) - \mathbf{A}(\boldsymbol{\varepsilon}_0,\theta_0) = \mathbf{I}, \quad c_1\mathbf{a}_1(\boldsymbol{\varepsilon}_0,\theta_0) + c_2\mathbf{a}_2(\boldsymbol{\varepsilon}_0,\theta_0) - \mathbf{a}(\boldsymbol{\varepsilon}_0,\theta_0) = \mathbf{0}, \qquad (34)$$

where $\mathbf{I}$ is the fourth order unit tensor. Finally, write $(30)_2$ for two-phase media as

$$\mathbf{l}(\boldsymbol{\varepsilon}_0,\theta_0) = c_1\mathbf{A}_1^{\mathrm{T}}(\boldsymbol{\varepsilon}_0,\theta_0)\mathbf{l}_1 + c_2\mathbf{A}_2^{\mathrm{T}}(\boldsymbol{\varepsilon}_0,\theta_0)\mathbf{l}_2. \qquad (35)$$

One can now solve for $\mathbf{A}_1^{\mathrm{T}}(\boldsymbol{\varepsilon}_0,\theta_0)$ and $\mathbf{A}_2^{\mathrm{T}}(\boldsymbol{\varepsilon}_0,\theta_0)$ from $(31)_1$ and $(34)_1$, and substitute them into (35) to find

$$\mathbf{l}(\boldsymbol{\varepsilon}_0,\theta_0) = \{\mathbf{L}(\boldsymbol{\varepsilon}_0,\theta_0) - \mathbf{L}_1\}(\mathbf{L}_2-\mathbf{L}_1)^{-1}(\mathbf{l}_2-\mathbf{l}_1)$$
$$+\mathbf{l}_1 + \mathbf{A}^{\mathrm{T}}(\boldsymbol{\varepsilon}_0,\theta_0)\{\mathbf{l}_1 - \mathbf{L}_1(\mathbf{L}_2-\mathbf{L}_1)^{-1}(\mathbf{l}_2-\mathbf{l}_1)\}. \qquad (36)$$

The diagonal symmetry of the $\mathbf{L}$ tensor has been invoked in the above derivation.

3.2. *Isotropic constituents with slipping interfaces*

We now consider a two-phase system which admits connections between mechanically and thermally induced pointwise fields that are not available in multiphase composites. The constituents are both isotropic, and the displacements of the interfaces are limited to nonlinear slip, i.e. $M_{nn} = 0$ in (9). Furthermore, the individual phases are assumed to contain no cracks or pores. In this particular system, the influence functions $\mathbf{a}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)$ and $\mathbf{d}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)$ are uniquely determined by their mechanical counterparts $\mathbf{A}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)$ and $\mathbf{D}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)$, respectively. Also, the general formula (36) can be reduced to a particularly convenient form. The specific results are:

$$\mathbf{a}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0) = \{\mathbf{I} - \mathbf{A}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)\}(\mathbf{L}_1-\mathbf{L}_2)^{-1}(\mathbf{l}_2-\mathbf{l}_1),$$
$$\mathbf{d}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0) = \{\mathbf{x} - \mathbf{D}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)\}(\mathbf{L}_1-\mathbf{L}_2)^{-1}(\mathbf{l}_2-\mathbf{l}_1),$$
$$\mathbf{a} = -\mathbf{A}(\mathbf{L}_1-\mathbf{L}_2)^{-1}(\mathbf{l}_2-\mathbf{l}_1), \quad \mathbf{l} = \mathbf{l}_1 + (\mathbf{L}-\mathbf{L}_1)(\mathbf{L}_2-\mathbf{L}_1)^{-1}(\mathbf{l}_2-\mathbf{l}_1). \qquad (37)$$

The validity of these relations will now be proved using the concept of incremental uniform fields in heterogeneous media introduced by Dvorak (1986). The composite is subjected to the boundary conditions (10), has the local fields (11), and the goal is to evaluate its response under a temperature increment $\mathrm{d}\theta_0$ from the current state, as in (14).

Superimpose on (10) the incremental loads $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$ and $\mathrm{d}\theta_0$:

$$\mathbf{u}(S) = \boldsymbol{\varepsilon}_0\mathbf{x} + \mathrm{d}\hat{\boldsymbol{\varepsilon}}\mathbf{x}, \quad \theta(S) = \theta_0 + \mathrm{d}\theta_0. \tag{38}$$

The $\mathrm{d}\theta_0$ is given but $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$ is not known; it is to be determined such that together with $\mathrm{d}\theta_0$ it creates a strain field $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$, and a stress field $\mathrm{d}\hat{\boldsymbol{\sigma}}$ which are both uniform in the entire representative volume. The desired magnitudes of $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$ and $\mathrm{d}\hat{\boldsymbol{\sigma}}$ can be readily determined from (1). Write the local incremental fields in both phases, make them equal, and evaluate the desired strain

$$\mathrm{d}\hat{\boldsymbol{\varepsilon}} = (\mathbf{L}_1 - \mathbf{L}_2)^{-1}(\mathbf{l}_2 - \mathbf{l}_1)\,\mathrm{d}\theta_0. \tag{39}$$

An analogous derivation (Dvorak, 1990) for a composite under overall uniform stress shows that a uniform stress field $\mathrm{d}\hat{\boldsymbol{\sigma}}$ can coexist with a temperature change $\mathrm{d}\theta_0$ if

$$\mathrm{d}\hat{\boldsymbol{\sigma}} = (\mathbf{M}_1 - \mathbf{M}_2)^{-1}(\mathbf{m}_2 - \mathbf{m}_1)\,\mathrm{d}\theta_0.$$

In the present system with isotropic constituents, both $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$ and $\mathrm{d}\hat{\boldsymbol{\sigma}}$ are hydrostatic, therefore, in the absence of normal interface displacements, the above increments cause only normal and continuous tractions at all interfaces. Of course, this also prevents interface slip, and the composite responds to the incremental loading (38) as if the interfaces were perfectly bonded.

To restore the original boundary conditions (12), the auxiliary strain $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$ must be removed. This is accomplished by changing (38) to

$$\mathbf{u}(S) = \boldsymbol{\varepsilon}^0\mathbf{x} + \mathrm{d}\hat{\boldsymbol{\varepsilon}}\mathbf{x} - \mathrm{d}\hat{\boldsymbol{\varepsilon}}\mathbf{x}, \quad \theta(S) = \theta_0 + \mathrm{d}\theta_0. \tag{40}$$

The incremental fields produced by the loading/unloading sequence (38) and (40) are

$$\mathrm{d}\boldsymbol{\varepsilon}_r(\mathbf{x}) = \mathrm{d}\hat{\boldsymbol{\varepsilon}} - \mathbf{A}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\hat{\boldsymbol{\varepsilon}}, \quad \mathrm{d}\mathbf{u}_r(\mathbf{x}) = \mathrm{d}\hat{\boldsymbol{\varepsilon}}\mathbf{x} - \mathbf{D}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)\,\mathrm{d}\hat{\boldsymbol{\varepsilon}}, \tag{41}$$

where $\mathrm{d}\hat{\boldsymbol{\varepsilon}}$ is to be substituted from (39). Note that (40) and (12) are identical, hence $\mathbf{a}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)$ and $\mathbf{d}_r(\mathbf{x};\boldsymbol{\varepsilon}_0,\theta_0)$ can be extracted by comparing (14) with (41). This leads to the expressions $(37)_1$ and $(37)_2$.

To recover $(37)_3$, recall that in the present derivation we rule out vacuous zones, hence

$$\mathbf{A} = \mathbf{F}_{12} + \mathbf{F}_{21}, \quad \mathbf{a} = \mathbf{f}_{12} + \mathbf{f}_{21}, \tag{42}$$

with $\mathbf{F}_{rs}$ and $\mathbf{f}_{rs}$ being given in (19). A substitution from $(37)_2$ to $(19)_2$, together with $(19)_1$ and (42), readily provides $(37)_3$.

Finally, a substitution of $(37)_1$ with $r = 2$, and of $(37)_3$ into $(31)_2$ gives

$$\mathbf{l}(\boldsymbol{\varepsilon}^0,\theta_0) = c_1\mathbf{l}_1 + c_2\mathbf{l}_2 + c_2(\mathbf{L}_2 - \mathbf{L}_1)(\mathbf{I} - \mathbf{A}_2)(\mathbf{L}_1 - \mathbf{L}_2)^{-1}(\mathbf{l}_2 - \mathbf{l}_1) - \mathbf{L}_1\mathbf{A}(\mathbf{L}_1 - \mathbf{L}_2)^{-1}(\mathbf{l}_2 - \mathbf{l}_1). \tag{43}$$

Solving for $\mathbf{A}_2$ in $(31)_1$ and substituting into (42) then provides $(37)_4$.

Recall that the thermal stress tensor $\mathbf{l}$ for two-phase composites with anisotropic constituents is given by $(31)_2$ or (36), and for systems with isotropic phases and slipping interfaces by $(37)_4$. These relations were arrived at by two entirely different approaches, hence it remains to be shown that they are equivalent under similar circumstances.

For isotropic constituents there is:

$$
\begin{aligned}
(l_1)_{ij} &= \alpha\delta_{ij},\\
(L_1)_{ijrs} &= \beta\delta_{ij}\delta_{rs}+\gamma(\delta_{ir}\delta_{js}+\delta_{is}\delta_{jr}-\tfrac{2}{3}\delta_{ij}\delta_{rs}),\\
(L_2-L_1)^{-1}_{rsmn} &= \xi\delta_{rs}\delta_{mn}+\zeta(\delta_{rm}\delta_{sn}+\delta_{rn}\delta_{sm}-\tfrac{2}{3}\delta_{rs}\delta_{mn}),\\
(l_2-l_1)_{mn} &= \lambda\delta_{mn},
\end{aligned}
\tag{44}
$$

where α, β, δ, ξ, ζ, λ are constants. Writing $\mathbf{A}^T$ in indicial notation and carrying out the summation in (36) according to (44) shows that the tensor $\mathbf{A}^T$ enters only as $(\mathbf{A}^T)_{pqii}$. Moreover, the continuity of normal displacements at S_{21}, which was assumed in the above derivation of $(37)_4$, implies that according to the definition of the $\mathbf{A}$ tensor, in (31)–(33), $A_{iikl}=0$, or in fact $(\mathbf{A}^T)_{pqii}=0$. This leads to the conclusion that (36) indeed reduces to the form $(37)_4$ when the phases are isotropic and the interfaces may experience only shear displacements.

Acknowledgements—Funding for this work was provided, in part, by the DARPA ONR HiTASC composites program at Rensselaer and by the Air Force Office of Scientific Research.

REFERENCES

Achenbach, J. D. and Zhu, H. (1989). Effect of interfacial zone on mechanical behavior and failure of fiber-reinforced composites. *J. Mech. Phys. Solids* **37**, 381–393.

Benveniste, Y. (1985). The effective mechanical behaviour of composite materials with imperfect contact between the constituents. *Mech. Mater.* **4**, 197–208.

Benveniste, Y. and Aboudi, J. (1984). A continuum model for fiber reinforced materials with debonding. *Int. J. Solids Structures* **20** (11/12), 935–951.

Benveniste, Y. and Dvorak, G. J. (1990a). On a correspondence between mechanical and thermal effects in two-phase composites. In Toshio Mura Anniversary Volume *Micromechanics and Inhomogeneity* (Edited by G. J. Weng, M. Taya and H. Abe), pp. 65–81. Springer, Berlin.

Benveniste, Y. and Dvorak, G. J. (1990b). On a correspondence between mechanical and thermal fields in composites with slipping interfaces. In *Inelastic Deformation of Composite Materials* (Edited by George J. Dvorak), IUTAM Symposium, Troy, NY, 29 May–1 June, 1990, pp. 77–98. Springer, Berlin.

Benveniste, Y. and Miloh, T. (1986). The effective conductivity of composites with imperfect thermal contact at constituent interfaces. *Int. J. Engng Sci.* **24**, 1537–1552.

Chen, I. W. and Argon, A. S. (1979a). Grain boundary and interphase boundary sliding in power law creep. *Acta Met.* **27**, 749–754.

Chen, I. W. and Argon, A. S. (1979b). Steady state power-law creep in heterogeneous alloys with coarse microstructures. *Acta Met.* **27**, 785–791.

Dvorak, G. J. (1983). Metal matrix composites: plasticity and fatigue. In *Mechanics of Composite Materials—Recent Advances* (Edited by Z. Hashin and C. T. Herakovich), pp. 73–92. Pergamon Press, New York.

Dvorak, G. J. (1986). Thermal expansion of elastic–plastic composite materials. *J. Appl. Mech.* **53**, 737–743.

Dvorak, G. J. (1990). On uniform fields in heterogeneous media. *Proc. R. Soc. London* **A431**, 89–110.

Dvorak, G. J. and Chen, T. (1989). Thermal expansion of three-phase composite materials. *J. Appl. Mech.* **56**, 418–422.

Hashin, Z., (1990). Thermoelastic properties of fiber composites with imperfect interface. *Mech. Mater.* **8**, 333–348.

Jasiuk, I., Mura, T. and Tsuchida, E. (1988). Thermal stresses and thermal expansion coefficients of short fiber composites with sliding interfaces. *J. Engng Mat. Tech.* **110**, 96–100.

Jasiuk, I. and Tong, Y. (1989). The effect of interface on the elastic stiffness of composites. In *Mechanics of Composite Materials and Structures* (Edited by J. N. Reddy and J. L. Teply), AMD-Vol. 100, pp. 49–54.

Lené, F. and Leguillon, D. (1982). Homogenized constitutive law for a partially cohesive composite material. *Int. J. Solids Structures* **18**, 443–458.

Levin, V. M. (1967). Thermal expansion coefficients of heterogeneous materials. *Mekhanika Tverdogo Tela*, **2**, 88–94. (English Translation *Mechanics of Solids*, **11**, 58–61.)

Mura, T., Jasiuk, I. and Tsuchida, E. (1985). The stress field of a sliding inclusion. *Int. J. Solids Structures* **12**, 1165–1179.

Tsuchida, E., Mura, T. and Dundurs, J. (1986). The elastic field of an elliptic inclusion with a slipping interface. *J. Appl. Mech.* **53**, pp. 103–107.

APPENDIX A

Equation (16) will be derived in this appendix. The considered multiphase composite may contain pores and cracks at arbitrary locations, but need not be matrix based, see Fig. A1 for a typical volume of such a composite. To derive eqn (16) it is sufficient to consider a three-phase composite as in Fig. A2. Note that phase "s" is in contact both with phases "r" and "p", a situation which would occur in non-matrix based composites. The notation in this figure is that described in Section 2.1. The derived average strain for the configuration of Fig. A2 can be readily generalized to multiphase composites of the type described in Fig. A1.

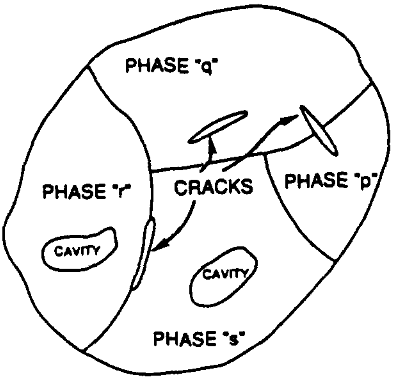

Fig. A1. A multiphase nonmatrix-based composite with defects.

We start by writing the average overall strain for the composite (Benveniste, 1985)

$$\bar{\varepsilon}_{ij} = \frac{1}{2V}\int_S (u_i n_j + u_j n_i)\,\mathrm{d}S, \tag{A1}$$

where S denotes the outside surface and **n** the outward normal to S. The average strain in phase r can be written as

$$\begin{aligned}\bar{\varepsilon}_{ij}^{(r)} = \frac{1}{2V_r}\int_{V_r}\left(\frac{\partial u_i^{(r)}}{\partial x_j} + \frac{\partial u_j^{(r)}}{\partial x_i}\right)\mathrm{d}V_r &= \frac{1}{2V_r}\int_S (u_i^{(r)} n_j + u_j^{(r)} n_i)\,\mathrm{d}S \\ &+ \frac{1}{2V_r}\int_{S_{rs}} (u_i^{(r)} n_j^{(rs)} + u_j^{(r)} n_i^{(rs)})\,\mathrm{d}S_{rs} + \frac{1}{2V_r}\int_{S_{rp}} (u_i^{(r)} n_j^{(rp)} + u_j^{(r)} n_i^{(rp)})\,\mathrm{d}S_{rp} \\ &+ \frac{1}{2V_r}\int_{S_{rv}} (u_i^{(r)} n_j^{(rv)} + u_j^{(r)} n_i^{(rv)})\,\mathrm{d}S_{rv}, \end{aligned} \tag{A2}$$

where Gauss's divergence theorem has been used.

Similarly, we can write the average strain in phases s and p as follows:

$$\bar{\varepsilon}_{ij}^{(s)} = \frac{1}{2V_s}\int_{S_{sr}} (u_i^{(s)} n_j^{(sr)} + u_j^{(s)} n_i^{(sr)})\,\mathrm{d}S_{sr} + \frac{1}{2V_s}\int_{S_{sp}} (u_i^{(s)} n_j^{(sp)} + u_j^{(s)} n_i^{(sp)})\,\mathrm{d}S_{sp},$$

$$\bar{\varepsilon}_{ij}^{(p)} = \frac{1}{2V_p}\int_{S_{pr}} (u_i^{(p)} n_j^{(pr)} + u_j^{(p)} n_i^{(pr)})\,\mathrm{d}S_{pr} + \frac{1}{2V_p}\int_{S_{ps}} (u_i^{(p)} n_j^{(ps)} + u_j^{(p)} n_i^{(ps)})\,\mathrm{d}S_{ps}, \tag{A3}$$

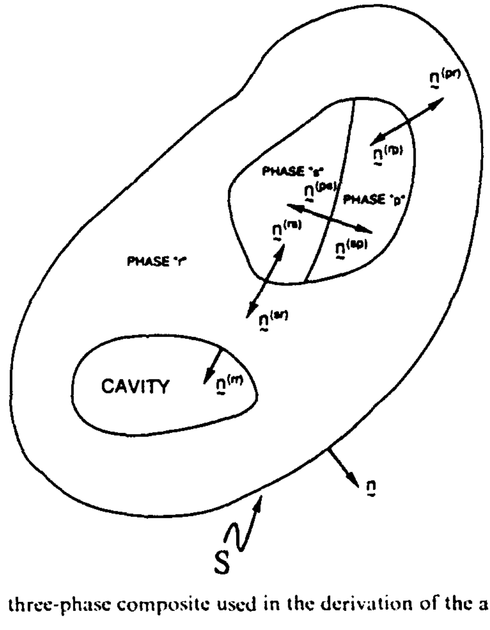

Fig. A2. A three-phase composite used in the derivation of the average strain.

where V_s and V_p denote the volumes of phases s and p, respectively. The total volume $V = V_r + V_s + V_p$. Multiplying eqn (A2) by $c_r = V_r/V$, and equation $(A3)_1$ and $(A3)_2$ by $c_s = V_s/V$ and $c_p = V_p/V$, respectively, and adding, results in

$$\bar{\varepsilon} = c_r\bar{\varepsilon}^{(r)} + c_s\bar{\varepsilon}^{(s)} + c_p\bar{\varepsilon}^{(p)} - \mathbf{J}^{(rs)} - \mathbf{J}^{(sr)} - \mathbf{J}^{(rp)} - \mathbf{J}^{(pr)} - \mathbf{J}^{(sp)} - \mathbf{J}^{(ps)} - \mathbf{J}^{(rr)}, \tag{A4}$$

where we used the definitions in (17). A generalization of (A4) to multiphase composites provides eqn (16). Equation (A4) or (16) reduces correctly to eqn (3) in Benveniste (1985) and (29) in Benveniste and Dvorak (1990b) which were written for two-phase matrix-based composites†. To draw a parallel with eqn (3) in Benveniste (1985), we simply note that **n** in that equation is given in the present notation by

$$\mathbf{n} = \mathbf{n}^{(21)} = -\mathbf{n}^{(12)}, \tag{A5}$$

where "1" denotes the matrix and "2" the inclusion and $[u_i]$ was defined as

$$[u_i]_{S_{12}} = u_i^{(2)} - u^{(1)} \quad \text{at } S_{12}, \tag{A6}$$

so that

$$\frac{1}{2V}\int_{S_{12}} ([u_i]n_j + [u_j]n_i)\,\mathrm{d}S_{12} = (\mathbf{J}_{21} + \mathbf{J}_{12})_{ij}. \tag{A7}$$

Recalling that no vacuous zones were present in the phases in these previous works, $\mathbf{J}_{11} = \mathbf{J}_{22} = \mathbf{0}$, and it is seen that eqn (3) in Benveniste (1985), and (29) in Benveniste and Dvorak (1990) are simply special cases of (A4).

APPENDIX B

An extension of the elastic reciprocal theorem to the situations in which the linearly elastic body contains interfaces of the type described in Section 2.1 can be written as

$$\int_V F'_i u''_i \,\mathrm{d}V + \int_S t'_i u''_i \,\mathrm{d}S + \int_{S_{rs}} t_i^{(sr)'} u_i^{(r)''}\,\mathrm{d}S_{rs} + \int_{S_{sr}} t_i^{(rs)'} u_i^{(s)''}\,\mathrm{d}S_{sr}$$
$$= \int_V F''_i u'_i \,\mathrm{d}V + \int_S t''_i u'_i \,\mathrm{d}S + \int_{S_{rs}} t_i^{(sr)''} u_i^{(r)'}\,\mathrm{d}S_{rs} + \int_{S_{sr}} t_i^{(rs)''} u_i^{(s)'}\,\mathrm{d}S_{sr}, \tag{B1}$$

where u''_i are the displacements caused by the system t''_i, F''_i, and u'_i are the displacements caused by the system t'_i, F'_i.

When distributions of eigenstresses $\lambda'_{ij} = l_{ij}\theta'$ and $\lambda''_{ij} = l_{ij}\theta''$ are respectively applied to the two systems, the local stress field is given by

$$\sigma'_{ij}(\mathbf{x}) = \tilde{\sigma}'_{ij}(\mathbf{x}) + \lambda'_{ij}, \tag{B2}$$

where

$$\tilde{\sigma}'_{ij}(\mathbf{x}) = L_{ijkl}(\mathbf{x})\varepsilon'_{kl}(\mathbf{x}). \tag{B3}$$

The field (B2) satisfies

$$\begin{aligned}
\tilde{\sigma}'_{ij,j} + F'_i + \lambda'_{ij,j} &= 0 && \text{in } V,\\
\tilde{\sigma}'_{ij}n_j + \lambda'_{ij}n_j &= t'_i && \text{on } S,\\
\tilde{\sigma}'_{ij}n_j^{(rs)} + \lambda'_{ij}n_j^{(rs)} &= t_i^{(sr)} && \text{on } S_{rs},\\
\tilde{\sigma}'_{ij}n_j^{(sr)} + \lambda'_{ij}n_j^{(sr)} &= t_i^{(rs)} && \text{on } S_{sr}.
\end{aligned} \tag{B4}$$

A similar representation holds for the double-primed system.

Define new body forces and surface tractions

$$\begin{aligned}
\tilde{F}_i &= F_i + \lambda_{ij,j}, & \tilde{t}_i &= t_i - \lambda_{ij}n_j,\\
\tilde{t}_i^{(sr)} &= t_i^{(sr)} - \lambda_{ij}n_j^{(rs)}, & \tilde{t}_i^{(rs)} &= t_i^{(rs)} - \lambda_{ij}n_j^{(sr)}
\end{aligned} \tag{B5}$$

and rewrite (B1) with F'_i, F''_i replaced by $\tilde{F}'_i$, $\tilde{F}''_i$, t'_i, t''_i by $\tilde{t}'_i$, $\tilde{t}''_i$ etc.

Consider first the right-hand side of (B1) rewritten as described above and substitute from (B4) to find

---

† Note that there is a misprint in the definition of **J** in eqn (30) in Benveniste and Dvorak (1990). The term V should be replaced by V_2 in that equation, as well as in eqns (36) and (47) of that work. Therefore, as we show in (A7), the correspondence between the **J** term in Benveniste and Dvorak (1990), and the $\mathbf{J}_{12}$, $\mathbf{J}_{21}$ terms in the present paper is: $c_2\mathbf{J} = (\mathbf{J}_{21} + \mathbf{J}_{12})$.

$$\int_V F''_i u'_i \,\mathrm{d}V + \int_V \lambda''_{ij,j} u'_i \,\mathrm{d}V + \int_S (t''_i - \lambda''_{ij} n_j) u'_i \,\mathrm{d}S - \int_{S_\nu} (t_i^{(\nu)''} - \lambda''_{ij} n_j^{(\nu)}) u_i^{(\nu)'} \,\mathrm{d}S_\nu + \int_{S_\psi} (t_i^{(\psi)''} - \lambda''_{ij} n_j^{(\psi)}) u_i^{(\psi)'} \,\mathrm{d}S_\psi . \qquad \text{(B6)}$$

Manipulating the second term in (B6) through the divergence theorem, one finds

$$\int_V \lambda''_{ij,j} u'_i \,\mathrm{d}V = \int_S \lambda''_{ij} u'_i n_j \,\mathrm{d}S + \int_{S_\nu} \lambda''_{ij} u_i^{(\nu)'} n_j^{(\nu)} \,\mathrm{d}S_\nu + \int_{S_\psi} \lambda''_{ij} u_i^{(\psi)'} n_j^{(\psi)} \,\mathrm{d}S_\psi - \int_V \lambda''_{ij} u'_{i,j} \,\mathrm{d}V . \qquad \text{(B7)}$$

Moreover, $u'_{i,j} = \varepsilon'_{ij} + \omega'_{ij}$, and since $\lambda''_{ij} = \lambda''_{ji}$, $\omega'_{ij} = -\omega'_{ji}$, it follows that when (B7) is substituted into (B6), some of the integrals on S and S_ν cancel out.

A similar procedure applied to the left-hand side of (B1) yields the form of the reciprocal theorem which is valid under internal defects at S_ν and in the presence of eigenstrains $\lambda_{ij} = l_{ij}\theta$:

$$\int_V F'_i u''_i \,\mathrm{d}V + \int_S t'_i u''_i \,\mathrm{d}S - \int_V \lambda'_{ij} \varepsilon''_{ij} \,\mathrm{d}V + \int_{S_\nu} t_i^{(\nu)'} u_i^{(\nu)''} \,\mathrm{d}S_\nu + \int_{S_\psi} t_i^{(\psi)'} u_i^{(\psi)''} \,\mathrm{d}S_\psi$$

$$= \int_V F''_i u'_i \,\mathrm{d}V + \int_S t''_i u'_i \,\mathrm{d}S - \int_V \lambda''_{ij} \varepsilon'_{ij} \,\mathrm{d}V + \int_{S_\nu} t_i^{(\nu)''} u_i^{(\nu)'} \,\mathrm{d}S_\nu + \int_{S_\psi} t_i^{(\psi)''} u_i^{(\psi)'} \,\mathrm{d}S_\psi . \qquad \text{(B8)}$$

It is interesting to note that although the linearity of the constitutive law in the phases has been assumed in (B1) and (B8), the constitutive law of the interfaces does not explicitly enter in these equations. In other words the relation between the interface tractions and the resulting interface displacements have not explicitly been used in (B8). We finally mention that eqn (B8) can also be used for an incremental set of loads ($\mathrm{d}F'_i$, $\mathrm{d}t'_i$, $\mathrm{d}\lambda'_{ij}$) and ($\mathrm{d}F''_i$, $\mathrm{d}t''_i$, $\mathrm{d}\lambda''_{ij}$) which are superimposed on an existing equilibrium state of deformation.